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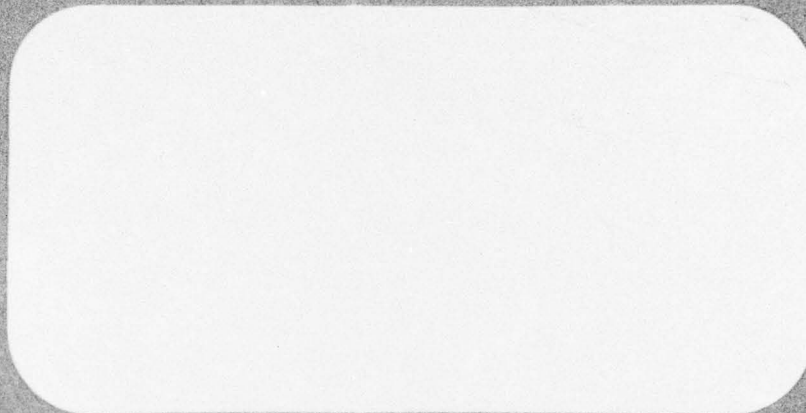UNCLASSIFIED   TR-311   NL

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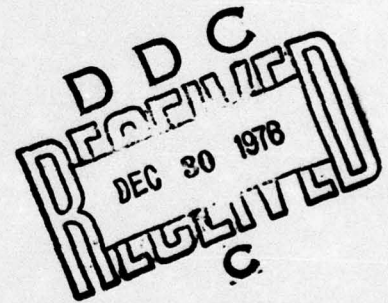TECHNICAL REPORT NO. 311

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THE INEVITABILITY OF THE PARADOX OF NEW MEMBERS

by

Joseph Raanan

1. Introduction.

The Paradox of New Members was introduced by Brams [3] and discussed further in Brams [4] and Brams and Affuso [5]. This Paradox refers to the un-expected phenomena that when a voting body is enlarged - by the addition of one or more new members - the power of some of the old members - as measured by some power indices - may increase rather than decrease.

One of the interesting questions raised by the Paradox of New Members is whether it is a property of the indices used to measure power (e.g., the Shapley-Shubik index [6], the Banzhaf index [2] , etc.), or whether it is an inherent property of voting bodies in general. That is, the question is whether there exists a power index that does not give rise to the paradox. In order to answer this question, we shall offer a set of reasonable axioms that any power index should satisfy, including one that explicitly rules out the Paradox of New Members. We shall then show that these axioms are incon-sistent, so that the paradox cannot be avoided if "power" is defined by an index satisfying these axioms. Another formulation of the Paradox of New Members is tested and it, too, results in inconsistency. Finally, we compare our axioms with those appearing in the literature and discuss the results and some of their implications.

2. The Axioms.

To fix ideas, we start by recalling some definitions and establishing some notation.

Definition 2.1: An n-person weighted majority voting game is a game $v = [q; w_1,\ldots,w_n]$ with $w_i > 0$, $i = 1,\ldots,n$, $\sum_{i=1}^{n} w_i = 1$ and $1/2 < q \leq 1$.

q is called the quota of the game and w_i is the weight of player $i = 1,\ldots,n$.

The set of players $\{1,2,\ldots,n\}$ will be denoted by N.

The set of all weighted majority voting games will be denoted by M.

<u>Definition 2.2</u>: A coalition $S \in 2^N$ in the weighted majority voting game v is <u>winning</u> if and only if $\sum_{i \in S} w_i \geq q$ and is <u>losing</u> otherwise.

<u>Definition 2.3</u>: A winning coalition $S \in 2^N$ is a <u>minimal winning</u> coalition if all coalitions $T \subsetneq S$ are losing.

<u>Definition 2.4</u>: A player $i \in N$ is called a <u>dictator</u> if $w_i \geq q$.

<u>Definition 2.5</u>: A player $i \in N$ is called a <u>veto-power player</u> if and only if $i \in \cap \{S \in 2^N | S$ is a minimal winning coalition$\}$.

<u>Definition 2.6</u>: The set of <u>imputations</u> for the game v, denoted by $A(v)$ is: $A(v) = \{x \in \mathbb{R}^n | x_i \geq 0, i = 1,\ldots,n$ and $\sum_{i=1}^{n} x_i = 1\}$.

<u>Definition 2.7</u>: The <u>core</u> of the game v, denoted by $C(v)$ is

$$C(v) = \{x \in A(v) | \sum_{i \in S} x_i \geq 1 \text{ for all winning coalitions } S \in 2^N\}.$$

<u>Definition 2.8</u>: A <u>power index</u> K is a function from M into $\mathbb{R}^n$, i.e., $K: M \rightarrow R^n$. $K_i(v)$ will denote the power index of player $i \in N$ in the game v.

Where no misunderstanding can result we shall write K and K_i instead of $K(v)$ and $K_i(v)$, respectively.

Let us note the following:

1. Since $q \leq 1 = \sum_{i \in N} w_i$, the <u>grand coalition</u>, N, is always winning and thus we always have winning and minimal winning coalitions.

2. A dictator is always a veto power player, but the opposite is not always true, as shown for example by the game $v = [3/4; 2/4, 1/4, 1/4]$ where player 1 is a veto power player, though not a dictator.

The axioms which a power index should satisfy are:

Axiom 1: $w_i = w_j \Rightarrow K_i(v) = K_j(v); \; w_i > w_j \Rightarrow K_i(v) \geq K_j(v).$

Axiom 2: $K_i(v) \geq 0 \; \forall \; i \in N$ and $\sum_{i \in N} K_i(v) = 1.$

Axiom 3: $w_i < q \iff K_i(v) < 1.$

Axiom 4: If $v' = [q; w_1', w_2', \ldots, w_n', w_{n+1}']$ and

$w_i' \leq w_i$ for $i = 1, \ldots, n$ then $K_i(v') \leq K_i(v)$

for $i = 1, \ldots, n.$

Axiom 1 can be called the symmetry or equity axiom and it says that players with the same weight have the same power and that power is a non-decreasing function of the weights.

Axiom 2 is just a normalization axiom, expressing the power as a discrete probability vector.

Axiom 3 is the Dictator axiom, saying that a player has all the power if and only if he determines single-handedly the outcomes of the voting. ($q > 1/2$ guarantees that we cannot have more than one dictator with $w_i \geq q$). In case we have a dictator, i, the only minimial winning coalition is $\{i\}$.

Axiom 4 was added in order to explicitly rule out the Paradox of New Members, in one of its forms (see section 4 for a discussion of another formulation.)

3. Proof of Inconsistency.

We shall now show that the axioms as written above are inconsistent, which implies that the Paradox of New Members in the form stated in axiom 4 cannot be avoided.

Theorem 3.1: No power index K, satisfying axioms 2,3, and 4 exists for the class of all weighted majority games.

Proof:

Let us consider the game $v = [q; q, w_2, \ldots, w_n]$, $n \geq 2$, with $q < 1$. Of course, $\sum_{i=2}^{n} w_i = 1 - q$.

Let v' be the game

$$v' = [q; w_1', w_2', \ldots, w_n', w_{n+1}'],$$

defined as follows:

$$w_i' = w_i, \quad i = 2, \ldots, n;$$

$$w_{n+1}' = q - \sum_{i=2}^{n} w_i = q - (1-q) = 2q - 1;$$

$$w_1' = w_1 - w_{n+1}' = q - (2q - 1) = 1 - q.$$

Clearly, the conditions of axiom 4 are satisfied. Also, by axiom 3, $K_1(v) = 1$, and $K_2(v) = K_3(v) = \ldots = K_n(v) = 0$. Note also that we have $w_i' < q$, $i = 1, \ldots, n+1$.

Now, let us construct a new game, v^2, out of v'

$$v^2 = [q; w_1^2, w_2^2]$$

where

$$w_1^2 = w_1' = 1 - q$$

and

$$w_2^2 = \sum_{i=2}^{n+1} w_i' = q.$$

By axiom 3, we have that $K_1(v^2) = 0$, $K_2(v^2) = 1$. Now, let us construct a sequence of games $v^3, v^4, \ldots, v^{n+1} = v'$ by the following procedure:

$$v^k = [q; w_1^k, w_2^k, \ldots, w_k^k], \quad k = 3, 4, \ldots, n+1,$$

where, for $k = 3, 4, \ldots, n+1$ we have:

$$w_1^k = w_1' = 1 - q,$$

$$w_i^k = w_i^{k-1}, \quad i = 2, \ldots, k-2,$$

$$w_{k-1}^{k} = w_{k-1}^{k-1} - w_{k}'$$

$$w_{k}^{k} = w_{k}'$$

Namely, we are "cutting" player 2 in a v^2 into its n "component" players, one at a time, while leaving player 1 intact. We continue this procedure, while conforming to the conditions of axiom 4 for each pair of games, v^j, v^{j+1}, $j = 2,\dots,n$, until finally we have, of course, $v^{n+1} = v'$.

One of the following must occur:

A. In one of the games $v^3, v^4, \dots, v^{n+1}$ we get $K_1(v^j) > 0$
   ($j = 3, \dots, n+1$) contradicting axiom 4 for the games v^{j-1} and v^j.

B. If A does not happen, then we must have

$$\sum_{i=2}^{n+1} K_i(v') = 1$$

and then either

(i) $K_{n+1}(v') = 1$, contradicting axiom 3

or

(ii) $K_{n+1}(v') < 1$ which implies that there exists m, $2 \leq m \leq n$ with $K_m(v') > 0$, contradictory axiom 4 for the games v and v'.

Since these cases are exhaustive and each of them leads to a contradiction we have shown that axioms 2,3 and 4 are inconsistent.

Q.E.D.

Theorem 3.1 shows that if one requires that the power index of the old member not increase if their weights do not increase when a new member is added then no such power index can be defined for the class of all weighted majority games. The next theorems strengthens this negative result.

Theorem 3.2: Let v be a weighted majority voting game, $v = [q; w_1, \ldots, w_n]$, and let K be a power index satisfying axioms 2,3 and 4. Then $K(v) \in C(v)$.

Proof:

Let $S_1, S_2, \ldots, S_\ell$ be the distinct minimal winning coalitions of v. Let $T = \bigcap_{j=1}^{\ell} S_j$.

Let $i \in S_2 \setminus S_1$. Then, by combining all members of S_1 into "one big player" we note that in the new game that big player has weight equal to $\sum_{m \in S_1} w_m \geq q$, so his power index becomes 1. Hence, when we "break up" that big player piece by piece into its original components - in the same way as we did it in theorem 3.1, we must have

$$\sum_{m \in S_1} K_m(v) = 1, \text{ and thus } K_i(v) = 0.$$

Using the same construction, we can show that if $j \in S_1 \setminus S_2$. We must have

$$\sum_{m \in S_2} K_m(v) = 1 \text{ and } K_j(v) = 0.$$

So, we know that the only way this can happen is if

$$\sum_{m \in S_1 \cap S_2} K_m(v) = 1.$$

Repeating this procedure for $S_3, \ldots, S_\ell$ we conclude that we must have

$$\sum_{m \in T} K_m(v) = 1.$$

But, it is well known that $C(v) = \{x \in A(v) \mid \sum_{i \in T} X_i = 1\}$

where $A(v) = \{x \in R^n | \sum_{i \in N} X_i = 1, x_i \geq 0 \ \forall \ i \in N\}$ = the set of imputations of the games v. So, $K(v) \in C(v)$.

Corollary 3.3: If $T = \phi$, no such K exists. I.e., for games with no veto power players, no power index satisfying axiom 2,3 and 4 exists.

Corollary 3.4: If $T = \{i\}$ for some $i \in N$ but $w_i < q$, then no power index satisfying axioms 2,3 and 4 exists for this game since by theorem 3.2, $K(v) \in C(v) = \{(0,\ldots,0,1,0,\ldots,0)\}$, but by axiom 3 $w_i < q \Rightarrow K_i(v) < 1$, a contradiction. I.e., if we have a veto-power player who is not a dictator, we get a contradiction.

4. Another Formulation of the Paradox - and Another Inconsistency.

The Paradox of New Members is not confined to the case when the absolute weights do not increase when a new member is added. It can also happen when the ratios of the weights of the odd members remain constant and the new player is added. For this case, axiom 4 should be replaced by

Axiom 4': If $v' = [q; w_1', w_2', \ldots, w_n', w_{n+1}']$ and $w_i' : w_j' = w_i : w_j$ for $i,j = 1,\ldots,n$ then $K_i(v') \leq K_i(v)$ for $i = 1,\ldots,n$.

But, in this case also we find that it is impossible to define a power index K satisfying axioms 1,2,3 and 4' on the class of all weighted majority games, as is shown by

Theorem 4.1: No power index K, satisfying axioms 1,2,3 and 4' can be defined on the class of all weighted majority games.

Proof:

Let v be the game $v = [q; q, w_2, \ldots, w_n]$, $n \geq 2$. Without loss of generality, let $w_2 \geq w_3 \geq \ldots, w_n > 0$. Then, by axiom 3, we get $K_1(v) = 1$, $K_i(v) = 0$, $i = 2, \ldots, n$.

Now, let v' be the game $v' = [q; \frac{q}{1+w_2}, \frac{w_2}{1+w_2}, \ldots, \frac{w_n}{1+w_2}, \frac{w_2}{1+w_2}]$.

Clearly, the conditions of axiom 4' are satisfied.

Now, in v', the weight of player 1 is $\frac{q}{1+w_2} < q$, so $K_1(v') < 1$, and by axiom 2, $\sum\limits_{i=2}^{n+1} K_i(v') > 0$.

By axiom 1, we know that $K_{n+1}(v') = K_2(v') \geq K_3(v') \geq \ldots \geq K_n(v')$. So, since the sum is strictly positive, we must have $K_2(v') > 0$, contradicting axiom 4'.

Q.E.D.

5. Discussion and Summary.

Axioms 1, 2, and 3 seem to be properties that every power index should satisfy. Allingham [1] suggests five axioms that should be satisfied by every power index on the class of all simple games, of which the weighted majority voting games are a subclass. We shall next show the correspondence of his axioms to ours, after introducing the following notation and definitions:

A simple game is a pair (N,W) with

N = the set of players = $\{1,2,\ldots,n\}$,

$W \subset 2^N - \emptyset$ is the set of winning coalitions, satisfying $(\forall T)$ $S \in W \Rightarrow S \cup T \in W$ (monotonicity).

A player is a dummy ($I \in D$) if and only if $S \in W \Rightarrow S \setminus \{i\} \in W$. (I.e., i does not belong to any minimal winning coalition.) Players i and j are symmetric, denoted $i \sim j$ if

$$(\forall S \,|\, i, j \notin S) \quad S \cup \{i\} \in W \Longleftrightarrow S \cup \{j\} \in W \,.$$

Then, a power index is defined by Allingham [1] to be a probability vector $\psi(N,W)$ on N, satisfying the following four axioms.

A.1 For every permutation π of N, $\psi_{\pi i}(N, \pi W) = \psi_i(N,W)$.

A.2 $i \in D \Longleftrightarrow \psi_i = 0$.

A.3 $i \sim j \Longleftrightarrow \psi_i = \psi_j$.

A.4 If (N,W) can be represented as weighted voting games then $w_i > w_j \Longrightarrow \psi_i \geq \psi_j$.

Our axiom 1 is just a combination of his axioms A.1 and A.4 for weighted majority games; axiom 2 is not listed as such but it is required that the power index be a discrete probability vector- exactly the contents of axiom 2; axiom 3, while not included in Allingham's list is implied by A.3 - i.e., it is weaker than A.3 - as lemma 5.1 below shows. Finally, A.3 can be deleted from the list because it is redundant - it is implied by A.1 (See appendix for proof.)

Lemma 5.1: Given axiom 2, axiom 3 is implied by Allingham's dummy axiom A.2 (I.e., A.2 $\Longrightarrow$ axiom 3).

Proof:

If there is a dictator $j \in N$ then the only minimal winning coalition is $\{j\}$. Hence all other players are dummies and, by the dummy axiom, we get that $K_i = 0 \; \forall \, j \neq i \in N$. So, since $\sum_{i \in N} K_i = 1$, we conclude that $K_j = 1$. That is, we showed that $w_j \geq q \Longrightarrow K_j = 1$ or that $K_j < 1 \Longrightarrow w_j < q$. To show the opposite implication we assume that there is no $j \in N$ who is a dictator. I.e., $w_i < q \; \forall \, i \in N$. Then, any minimal winning coalition must consist of at least two players and, since we now have at least two non-dummies, both must get, by A.2, positive power indices. So, there can be no player $i \in N$ with $K_i = 1$. Hence, we conclude that $w_i < q \Longrightarrow K_i < 1$.

Q.E.D.

Axioms 4 and 4' were explicitly introduced - in addition to axioms 1,2, and 3 that every power index should satisfy - in order to prevent the Paradox of New Members from occurring. However, as we have shown above, it leads to inconsistencies and, if we indeed take axioms 1,2 and 3 to be ones that must be satisfied by every power index, we must discard axioms 4 and 4' and conclude that the Paradox of New Members can not, in general, be avoided. It apparently stems from our incomplete understanding of the structure and behavior of voting bodies and not from shortcomings of any power index.

In conclusion, we would like to add two remarks:

1. Axiom 4 seems to impose stronger conditions than axiom 4' and, indeed, it is enough to have it and axioms 2 and 3 in order to prove inconsistency, while the proof of inconsistency when we use axiom 4' does require the use of axiom 1, in addition to axioms 2 and 3.

2. It is indeed surprising to find out that both the Shapely-Shubik index and the Banzhaf index exhibit the Paradox of New Members, as the following example shows:

$$v = [5/8; \; 3/7, 2/7, 2/7], \quad v' = [5/8; \; 3/8, 2/8, 2/8, 1/8] \; .$$

For v the value of the Shapely-Shubik index is $\phi(v) = (4/6, 1/6, 1/6)$ and the value of the Banzhaf index is $\beta(v) = (3/5, 1/5, 1/5)$, while for v' we have $\phi(v') = \beta(v') = (5/12, 3/12, 3/12, 1/12)$.

Thus even though v and v' satisfy the conditions of both axiom 4 and 4', we get that players 2 and 3 increase their power, using both indices, when the new member is added. (This example is due to Brams [2,3], basically, with the only difference being that we normalize the weights whereas he does not.)

However, the above mentioned example is now less surprising because we know that no power index exists that is invulnerable to the Paradox of New Members. Note also that the game v above indeed has a veto power player who is not a dictator, and we know that such games admit no power index that is invulnerable to the Paradox - by corollary 3.4.

6. Appendix.

We prove here the following

Lemma 6.1:   A.1 $\Rightarrow$ A.3.

Proof:

Let σ be the permutation that interchanges i and j and leaves all other players the same. I.e.,

$$\sigma(k) = \begin{cases} k & \text{if } k \neq i, \ k \neq j \\ i & \text{if } k = j \\ j & \text{if } k = i \end{cases}$$

Let $i \sim j$. Then $\sigma W = W$. Now, by A.1 we have $\psi_i(N,W) = \psi_{\sigma i}(N,\sigma W) = \psi_j(N,\sigma W) = \psi_j(N,W)$ so $\psi_i = \psi_j$.

On the other hand, if $\psi_i = \psi_j$ then

$$\psi_j(N,W) = \psi_i(N,N) = \psi_{\sigma i}(N,\sigma W) = \psi_j(N,\sigma W)$$

which implies that $W = \sigma W$ or that $i \sim j$.

Q.E.D.

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This paper presents an axiomatic proof that the Paradox of New Members can not be avoided. Axioms that should be satisfied by every power index are offered and, when an axiom specifically intended to avoid the Paradox is added, a contradiction is derived. This is done for the two different formulation of the Paradox of New Members.